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Periodic heat transfer by forced laminar boundary layer flow over a semi-infinite flat plate

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Abstract

The paper reports a study of periodic convection in a steady forced laminar boundary layer flow over a semi-infinite impermeable flat plate due to periodical variation of the wall heat flux. The Fourier transform based approach allows to obtain a transfer function for the boundary layer that can be used to solve also transient (non-periodic) heating problems, and examples are reported comparing with available studies in the open literature. The effect of periodic heating on the value of the the average heat transfer coefficient is analysed and it is found to be important for relatively high frequency fluctuations of the imposed heat flux, whereas fluctuation amplitude of the instantaneous heat transfer coefficient is non-negligible also for lower exciting frequency.

1. Introduction

In the field of heat transfer technology, the unsteady forced convection represents a main topic as many thermal systems (like heat exchangers, turbomachines etc.) are often subjected to time variations of thermal boundary conditions. Unsteady forced convection in laminar external flows is of fundamental importance in many aspects of practical engineering and modern technology, as witnessed by the large amount of theoretical and experimental studies on this subject. Among others, the case of steady momentum boundary layer flows subject to variable thermal boundary conditions is of great importance for the implications to heat transfer problems. These problems can be classified as transient or periodic, with respect to the dependence on time of the boundary conditions, and the first ones have certainly been the most investigated. Several studies can be found in the open literature concerning the solution of the unsteady energy equation for laminar external flow with step changes in the wall temperature, for example Riley [1], Chao and Cheema [2], Dennis [3], Van Dyke [4], studied the problem of the transient heat transfer initiated by a step change in the temperature (uniformly distributed) of the plate over which a fluid is flowing. More recently the transient heat transfer due to the generation of an impulsive heat flux step change on the upper face of the flat plate was studied and several works dealing with this problem can be found [5–11]. Studies about periodic problems are less common, despite of the importance in many advanced applications like, for example, cooling of satellite components, solar energy transformation etc. but also as methods to measure heat transfer coefficient [12]. There is a trend to extrapolate the use of the heat transfer coefficient to transient regimes as well and all the experimental evaluations of this coefficient with transient methods rely on the assumption that heat

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Nomenclature

d, D	thickness, $d = \frac{D}{L}$
f	non-dimensional stream function
h	convective heat transfer coefficient
k	thermal conductivity
L	characteristic length
P	free stream-wall temperature difference
Pr	Prandtl number
q	heat flux
Re_L	Reynolds number
T	temperature
$\tilde{t}, t$	time, $t = \frac{\tilde{t}U_\infty}{L}$
$\tilde{u}, u$	velocity along the plate, $u = \frac{\tilde{u}}{U_\infty}$
U_∞	free stream velocity
$\tilde{v}, v$	velocity normal to the plate, $v = \frac{\tilde{v}}{U_\infty}$
$\tilde{x}, x$	Cartesian coordinate along the plate, $x = \frac{\tilde{x}}{L}$
$\tilde{y}, y$	Cartesian coordinate normal to the plate, $y = \frac{\tilde{y}}{L}$

Greek symbols

α	thermal diffusivity
γ	phase delay
η, ξ	non-dimensional coordinates
Θ, θ, θ^*	non-dimensional temperatures
$\tau, \hat{\tau}$	non-dimensional times, $\tau = \frac{\tilde{t}U_\infty}{\tilde{x}} = \frac{t}{x}$; $\hat{\tau} = \tau Pr^{-1/3}$
Φ	non-dimensional temperature gradient at the wall surface
ψ	stream function

Subscripts

a	time averaged
i, r	imaginary and real parts of complex number
st	steady
w	wall surface
∞	free stream

transfer coefficient is not influenced by the temperature unsteadiness (see for example [12,13]), despite of the existing experimental [14] and theoretical [15,16] evidence that in transient methods the heat transfer coefficient varies with time. The purpose of the present investigation is to determine the evolution of the heat transfer process for incompressible forced laminar flow over a semi-infinite plate subjected to periodic variation of the wall heat flux. It is assumed that the periodic regime is always reached. This approach allows to define a sort of boundary layer transfer function that can be used to deal also with unsteady non-periodic boundary conditions and proper comparison with available results are reported. In the author knowledge, this situation has not been previously treated in the literature.

2. Basic equations

Under constant properties conditions and with zero pressure gradient, the similarity solution approach to mass and momentum conservation equations, for the steady laminar boundary layer over a semi-infinite flat plate, leads to the well known equation (see for example [17]) for the non-dimensional stream function $f(\eta) = \psi Re_L^{1/2} x^{-1/2}$:

$$\begin{aligned} & f_{\eta\eta\eta} + \frac{1}{2} f_{\eta\eta} f = 0 \\ & f_\eta(0) = 0; \quad f(0) = 0; \quad f_\eta(\eta \to \infty) = 1 \end{aligned} \tag{1}$$

where $\eta = yx^{-1/2}Re_L^{1/2}$ and $x = \frac{\tilde{x}}{L}$; $y = \frac{\tilde{y}}{L}$. The time-dependent energy equation for the boundary layer is then

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{Re_L Pr}\frac{\partial^2 T}{\partial y^2} \tag{2}$$

where $u = \frac{\tilde{u}}{U_\infty}$; $v = \frac{\tilde{v}}{U_\infty}$, the non-dimensional time is $t = \frac{\tilde{t}U_\infty}{L}$ and the dissipative term was neglected. After the coordinate transformation: $(\eta, \xi) = (yx^{-1/2}Re_L^{1/2}, x)$, Eq. (2) becomes

$$\xi Pr\frac{\partial T}{\partial t} + \xi Pr f_\eta T_\xi - \frac{Pr}{2} f T_\eta = T_{\eta\eta} \tag{3}$$

On the wall, a time varying heat flux is imposed. Assuming that the variation takes place simultaneously at each location along the plate, so that heat flux is uniform along the wall, the boundary conditions become

$$-k\left(\frac{\partial T}{\partial \tilde{y}}\right)_{y=0} = q_w(t); \quad T(\xi, \infty, t) = T_\infty \tag{4}$$

where the function $q_w(t)$ will be considered periodic of arbitrary shape and T_∞ is the free stream temperature.

3. The periodic convection

The assumption of periodic time variation allows to split the temperature and flux fields into steady and time varying components. Define the time-average operator $\langle\,\rangle$ as

$$g_a = \langle g \rangle = \lim_{t_0 \to \infty} \frac{1}{t_0} \int_0^{t_0} g(t)\,dt \tag{5}$$

and the Fourier transforms of the temperature and heat flux fluctuating parts:

$$T(\xi,\eta,t) = T_a(\xi,\eta) + \int_{-\infty}^{\infty} S(\xi,\eta,\omega)e^{i\omega t}\,d\omega;$$
$$q_w(t) = q_{w,a} + \int_{-\infty}^{\infty} \widetilde{\Omega}(\omega)e^{i\omega t}\,d\omega \tag{6}$$

with $\omega = \frac{\tilde{\omega}L}{U_\infty} = \frac{2\pi \mathrm{fr} L}{U_\infty}$ where fr is the frequency. This allows to separate the steady problem from the time-dependent one.

3.1. The steady problem

Applying the time-average operator $\langle\,\rangle$ to Eq. (3) and to the boundary conditions (4) yields

$$\xi Pr f_\eta \frac{\partial T_a}{\partial \xi} = \frac{\partial^2 T_a}{\partial \eta^2} + \frac{Pr}{2} f \frac{\partial T_a}{\partial \eta} \tag{7}$$
$$q_{w,a} = -\frac{k}{L}\left(\frac{\partial T_a}{\partial \eta}\right)_{\eta=0} \xi^{-1/2} Re_L^{1/2} \tag{8}$$

It is well known that the steady uniform heat flux problem admits similarity solutions [17] (also under a wider range of velocity distribution in the free flow, see for example [19,20]) under the form:

$$T_a = P(\xi)\Theta_{st}(\eta); \quad P(\xi) = A\xi^{1/2} = T_a(\xi,0) - T_\infty$$

where A is a constant. In fact, Eqs. (7) and (8) become

$$\frac{\partial^2 \Theta_{st}}{\partial \eta^2} + \frac{Pr}{2} f \frac{\partial \Theta_{st}}{\partial \eta} - \frac{Pr}{2} f_\eta \Theta_{st} = 0 \tag{9}$$
$$q_{w,a} = \langle q_w(t)\rangle = \frac{k}{L} A \Phi_{st} Re_L^{1/2}$$

with boundary conditions: $\Theta_{st}(0) = 1$; $\Theta_{st}(\infty) = 0$, and $\Phi_{st} = -\left(\frac{\partial \Theta_{st}}{\partial \eta}\right)_{\eta=0}$. The numerical solution of Eq. (9) by a fourth order Runge–Kutta method for different Prandtl numbers confirmed that the approximation $\Phi_{st} = -\left(\frac{\partial \Theta_{st}}{\partial \eta}\right)_{\eta=0} = 0.459 Pr^{1/3}$ holds for $Pr > 0.6$, in accordance with [21].

3.2. The unsteady problem

From Eqs. (3), (6), (7) and (8) the following equations are found:

$$i\xi Pr\omega S + \xi Pr f_\eta \frac{\partial S}{\partial \xi} = \frac{\partial^2 S}{\partial \eta^2} + \frac{Pr}{2} f \frac{\partial S}{\partial \eta} \tag{10}$$
$$S(\xi,\infty,\omega) = 0; \quad S(0,\eta,\omega) = 0;$$
$$\Omega(\omega) = -\xi^{-1/2}\left(\frac{\partial S}{\partial \eta}\right)_{\eta=0} \tag{11}$$

where

$$\Omega = \widetilde{\Omega}\frac{L}{k} Re_L^{-1/2} \tag{12}$$

The solution of the problem set by Eqs. (10) and (11) yields the boundary layer temperature fluctuation under harmonic forcing at any frequency, then the use of Eq. (6) allows to reconstruct the fluctuation under non-harmonic input conditions.

4. Solutions of the periodic problem

Considering that Ω is independent of position and introducing the variables: $G = \omega^{1/2}\frac{S}{\Omega}$; $w = \xi\omega$, Eqs. (10) and (11) can be reduced to a more useful form

$$iPrwG + wPrf_\eta \frac{\partial G}{\partial w} = \frac{\partial^2 G}{\partial \eta^2} + \frac{Pr}{2} f \frac{\partial G}{\partial \eta} \tag{13}$$
$$G(\xi,\infty,\omega) = 0; \quad G(0,\eta,\omega) = 0; \quad \left(\frac{\partial G}{\partial \eta}\right)_{\eta=0} = -w^{1/2} \tag{14}$$

showing that the functional dependence of G is actually: $G = G(\omega\xi,\eta,Pr) = G(w,\eta,Pr)$. Consider now, without loss of generality, the following decomposition of the complex field G

$$G = G_0(w,Pr)\Theta(w,\eta,Pr)$$

with $\Theta(w,0,Pr) = 1$; $\Theta(w,\infty,Pr) = 0$ and $G_0(w,Pr) = G(w,0,Pr)$. Then Eq. (13) becomes

$$iPrwG_0\Theta + Prwf_\eta \frac{\partial G_0}{\partial w}\Theta + Prwf_\eta \frac{\partial \Theta}{\partial w} G_0 = G_0\left[\frac{\partial^2 \Theta}{\partial \eta^2} + \frac{Pr}{2} f \frac{\partial \Theta}{\partial \eta}\right] \tag{15}$$

with the condition on G_0 given by

$$G_0(\xi,\omega) = \frac{w^{1/2}}{\Phi(w,Pr)} \tag{16}$$

where $\Phi(w,Pr) = -\left(\frac{\partial \Theta}{\partial \eta}\right)_{\eta=0}$ is the non-dimensional temperature gradient at the plate surface.

4.1. The quasi-steady boundary layer

Consider now the limit for $\omega \to 0$, i.e., slow time variation of all variables. A solution $\Theta(\eta)$ of Eq. (15) independent of w (that implies $\Phi = -\left(\frac{\partial \Theta}{\partial \eta}\right)_{\eta=0}$ independent of w as well) can be found setting $G_0(w) = \frac{w^{1/2}}{\Phi(Pr)}$, then

$$\frac{\partial^2 \Theta}{\partial \eta^2} + \frac{Pr}{2} f \frac{\partial \Theta}{\partial \eta} = f_\eta \frac{Pr}{2}\Theta \qquad \Theta(0) = 1; \quad \Theta(\infty) = 0 \tag{17}$$

that coincides with the steady problem with uniform heat flux condition (see Eq. (9)). Φ is now a real function of Pr only and it is equal to the value found for the steady case. The asymptotic solution for $\omega \to 0$ is then

$$G(w,\eta,Pr) = G_0(w,Pr)\Theta(\eta,Pr) = \frac{w^{1/2}}{\Phi_{st}(Pr)}\Theta_{st}(\eta,Pr) \tag{18}$$

It is interesting to observe that for the harmonic case (under the above mentioned assumptions):

$$T(0,t) - T_\infty = T_a - T_\infty + T'(t) = T_a - T_\infty + Re\{G_0 \Omega \omega^{-1/2} e^{i\omega t}\} \tag{19}$$

$$q_w(t) = q_{w,a} + q'_w(t) = q_{w,a} + \frac{k}{L} \xi^{-1/2} Re_L^{1/2} \Phi \times Re\{G_0 \Omega \omega^{-1/2} e^{i\omega t}\} \tag{20}$$

then the fluctuating parts of $q_w(t)$ and $T_w(t) - T_\infty$ are in phase and the ratio

$$h = \frac{q'_w(t)}{T'(0,t)} = \frac{k}{L} \xi^{-1/2} Re_L^{1/2} \Phi = h_{st} \tag{21}$$

is equal to the value found for the steady case.

4.2. Numerical results

Eq. (10) was re-written in vectorial form as

$$\frac{\partial^2 \mathbf{G}}{\partial \eta^2} + \frac{Pr}{2} f(\eta) \frac{\partial \mathbf{G}}{\partial \eta} = Prwf_\eta \frac{\partial \mathbf{G}}{\partial w} + wPr\widehat{\mathbf{C}}\mathbf{G}$$

$$\mathbf{G}(\xi, \infty, \omega) = 0; \quad \mathbf{G}(0, \eta, \omega) = 0; \quad \left(\frac{\partial \mathbf{G}}{\partial \eta}\right)_{\eta=0} = -w^{1/2}$$

with

$$\mathbf{G} = \begin{pmatrix} G_r \\ G_i \end{pmatrix}; \quad \widehat{\mathbf{C}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and solved numerically (for Prandtl number ranging from 0.1 to 15) using a finite difference scheme with local grid refinement to better catch the steepest variations. The numerical solution of Eq. (1), by a fourth order Runge–Kutta method and following a procedure similar to that described in [18], provided the values of the functions $f(\eta)$ and $f_\eta(\eta)$ at the nodes. Fig. 1(a) shows the function $|G(w, 0, Pr)| = |G_0(w, Pr)|$ for different values of Pr. By the transformation: $\lambda = wPr^{1/3}$; $F_0(\lambda) = Pr^{1/2} G_0(\lambda, Pr)$, all the data (for $Pr > 0.6$) collapse on a single curve as shown in Fig. 1(b); only the case for $Pr = 0.1$ shows a different behaviour. The asymptotic solution for $\omega \to 0$ (18), written in terms of $F_0(\lambda)$, yields

$$|F_0| = Pr^{1/2}|G_0| = Pr^{1/2} \frac{w^{1/2}}{\Phi_{st}} = \frac{\sqrt{\lambda}}{\Phi_{st} Pr^{-1/3}}$$

As above mentioned, $\Phi_{st} = 0.459 Pr^{1/3}$ for $Pr > 0.6$, then the asymptotic form of $F_0(\lambda)$ is: $|F_0(\lambda)| = 2.17\lambda^{1/2}$ and Fig. 1(b) shows this result, and also that $\lim_{\lambda \to \infty} |F_0(\lambda)| = F_\infty = \text{const}$. It is of certain interest to consider the analogy with a pure conduction problem (see Appendix A for details) obtained by substituting the fluid with a thin layer of a solid material ($u, v = 0$) having thickness proportional to the boundary layer thickness and the same properties of the fluid. In this case the "transfer function" G_c linking the wall temperature fluctuation to the imposed heat flux can be calculated analytically

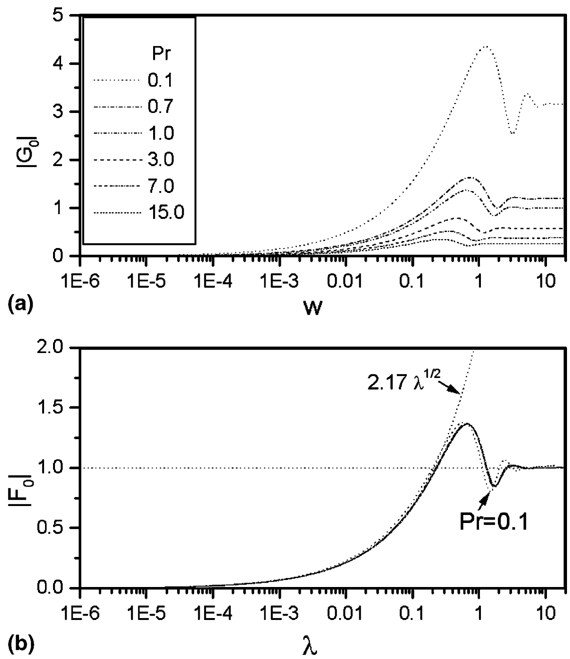

Fig. 1. (a) Modulus of the function $|G_0|$ vs the non-dimensional variable $w = \omega\xi$ for different values of the Prandtl number. (b) Modulus of the transfer function $|F_0|$ vs the nondimensional variable $\lambda = \omega\xi Pr^{1/3}$ for different values of Prandtl number.

(see Appendix A). Comparing the asymptotic behaviour when $\lambda \to 0$ with the results obtained for the boundary layer, shows that the limit $\lim_{\lambda \to 0} |F_0(\lambda)| = \frac{\lambda^{1/2}}{m}$ holds for both cases (choosing $m = 0.459$) and also the limit $\lim_{\lambda \to 0} |F_0(\lambda)| = F_\infty = 1$ holds for both cases. This may be appreciated in Fig. 2 where the results for the convective and conductive case are compared having used $\Phi_{c,st} = \Phi_{st}$. The convective terms tend to become more

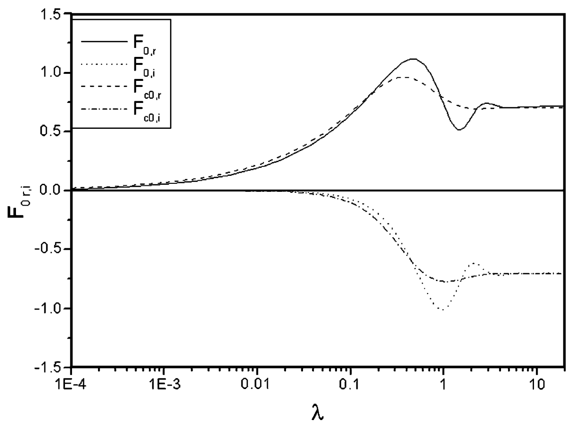

Fig. 2. Comparison between the imaginary and real part of the function F_0 and the equivalent functions for the conductive case.

effective for intermediate exciting frequencies. The phase delay between the imposed heat flux fluctuation and the wall-fluid temperature difference can be calculated by setting: $S_0 = S(\xi, 0, \omega) = |S_0|e^{i(\zeta+\gamma)}$ and $\Omega = |\Omega|e^{i\zeta}$ where γ is the phase delay. Then

$$\tan(\gamma) = \frac{\mathrm{Im}\{G_0\}}{\mathrm{Re}\{G_0\}} = -\frac{\mathrm{Im}\{\Phi\}}{\mathrm{Re}\{\Phi\}}$$

with $G_0 = \frac{w^{1/2}}{\Phi(\xi,\omega)}$, showing that the phase delay disappears when $\omega \to 0$. A comparison with the conductive case shows again a common asymptotic behaviour for $\omega \to 0$ and $\omega \to \infty$.

5. Heat transfer coefficient for periodic heating

When transient methods are used to estimate the heat transfer coefficient, the hypothesis of constant coefficient is almost universally used, despite of the fact that experimental [14] and theoretical [15,16] results have evidenced the effect of temperature field unsteadiness on the heat transfer coefficient. By definition

$$h = \frac{q_w}{(T_s - T_\infty)} = \frac{q_{w,a} + q'_w(t)}{(T_{s,a} - T_\infty) + T'_w(t)} \tag{22}$$

and for the harmonic case $T'_w(0,t) = Re\{S_0 e^{i\omega t}\}$; $q'_w(t) = \frac{h_{st}}{\Phi_{st}} Re\{S_0 \Phi e^{i\omega t}\}$ (from (19)–(21)). It is then easy to show that the deviation of the heat transfer coefficient from the steady state value $h'' = h - h_{st} = h' + (h_a - h_{st})$ is

$$h'' = h_{st}\Phi_{st}^{-1}\frac{\sin(\omega t')(\Phi_r - \Phi_{st}) + \cos(\omega t')\Phi_i}{s^{-1} + \sin(\omega t')} \tag{23}$$

where $s = \frac{|S_0|}{(T_{w,a} - T_\infty)} < 1$ (to avoid h'' becoming infinite, then loosing its physical meaning), $t' = t + t_0$ and t_0 is defined by $e^{-i\omega t_0} = \frac{S_0}{|S_0|}$. It should be noticed that $h_a = \langle h \rangle$ is not necessarily equal to h_{st} (the value found for the steady problem) in fact

$$h_a = \langle h'' \rangle + h_{st} = h_{st}\left(\frac{\Phi_r}{\Phi_{st}} - \frac{(\Phi_r - \Phi_{st})}{\Phi_{st}}\frac{1}{\sqrt{1-s^2}}\right);$$

$$\langle h'' \rangle = h_{st}\frac{(\Phi_r - \Phi_{st})}{\Phi_{st}}C(s)$$

with $C(s) = 1 - \frac{1}{\sqrt{1-s^2}}$. The values of $\frac{\langle h'' \rangle}{h_{st}}$ are shown in Fig. 3(a) (from the numerically calculated values of Φ) as a function of λ, showing that the deviation becomes important only for relatively large frequencies. The fluctuation amplitude of h''

$$\sigma^2 = \langle h''^2 \rangle - \langle h'' \rangle^2$$
$$= -\frac{h_{st}^2}{\Phi_{st}^2}\left\{(\Phi_r - \Phi_{st})^2\left[C(s) - \frac{s^2}{1-s^2}\right] + \Phi_i^2 C(s)\right\}$$

may become important also for lower frequencies (a sample of the results is reported in Fig. 3(b) for $s = 0.1$–0.5).

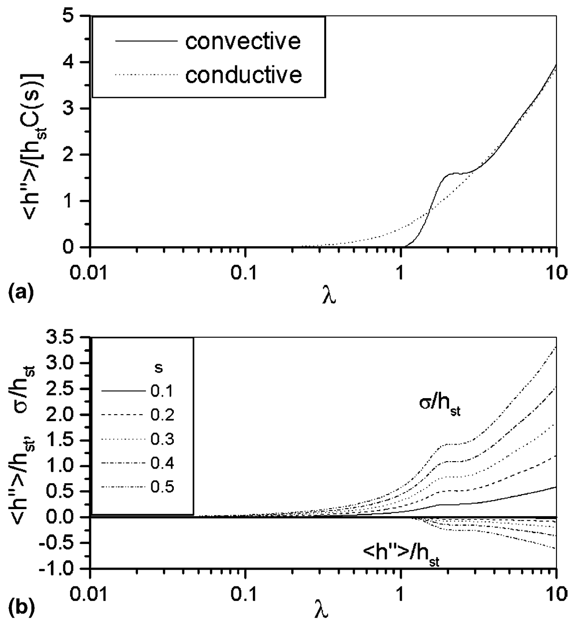

Fig. 3. (a) Relative deviation of the heat transfer coefficient from its steady value as a function of the nondimensional variable $\lambda = \omega\xi Pr^{1/3}$ (for $Pr > 0.6$). (b) Values of $\langle h'' \rangle / h_{st}$ and σ / h_{st} as a function of the non-dimensional variable $\lambda = \omega\xi Pr^{1/3}$ for different temperature fluctuation amplitudes ($s = 0.1$–0.5).

6. Transfer function approach to transient heating

Once the function G_0 is known, it is possible to evaluate the wall surface temperature evolution from the heat flux time variation and also impulsive heating problems can be analysed with sufficient approximation. Consider as a first example the case treated in [15], the plate is initially at a given temperature $T_i = T_\infty$ with no heat transfer, suddenly the heat flux become different from zero and remains constant for the rest of the time, but with uniform distribution along the plate. This problem can be approximated by a periodic one, choosing a periodic function for the heat flux time history made by a repetition of a heating time interval (τ_1) followed by a non-heating time interval (τ_2). If the time interval τ_2 is long enough, the plate temperature will have the time to reach a value very close to T_i before the heat flux is switched on again. Under this hypothesis, the problem can be solved by using the calculated function G_0 through the relation

$$T'_w(\xi, t) = \int_{-\infty}^{\infty} S_0(\xi, \omega)e^{i\omega t}\,d\omega$$
$$= \int_{-\infty}^{\infty} G_0(\xi, \omega)\Omega(\omega)\omega^{-1/2}e^{i\omega t}\,d\omega \tag{24}$$

where $\widetilde{\Omega}(\omega) = \frac{k}{L}Re_L^{1/2}\Omega$ is the Fourier transform of the heat flux time history. From the analysis reported in Appendix B, it is possible to show that the results found by [15] for the two cases $Pr = 0.72$ and $Pr = 6.7$ can actually be expressed in a unique form introducing the transformation:

$$\theta^* = \theta Pr^{1/3}; \quad \hat{\tau} = \tau Pr^{-1/3}$$

where the non-dimensional temperature θ and the non-dimensional time τ are defined (following [15]) as

$$\tau = \frac{U\tilde{t}}{\tilde{x}}; \quad \theta = \frac{T'}{\frac{q_{\max}}{k}\left(\frac{v\tilde{x}}{U_\infty}\right)^{1/2}}$$

It is worth to notice that the present approach allows to show that, for $Pr > 0.6$, all the impulsive heating problem of this kind can be simulated by a single numerical experiment obtained, say, with $Pr = 1$. The wall temperature variation was calculated following this procedure and compared to the results of [15] in Fig. 4. The agreement between the two computations is quite good and the discrepancy is lower than 0.5%. To be noticed that the time intervals τ_1 and τ_2 where chosen so to reach a non-dimensional temperature, before heat flux switching on, lower than 0.5% its maximum value. Interestingly enough, the same function G_0 previously calculated was also used to predict the results obtained by Lachi et al. [16] under different heating conditions. In that case, the heat flux changes from zero to a $10\ \mathrm{W/m^2}$ and it is maintained to that value for a given time interval (0.3 s) and then it is again suddenly increased to $100\ \mathrm{W/m^2}$. Again this problem can be approximated by a periodic one by choosing two large enough time intervals Δt_1 (for the non-heating period) and Δt_2 (for the $100\ \mathrm{W/m^2}$ heating period). The wall temperature variation can be still obtained through Eq. (24) after calculating the Fourier transform of the imposed heat flux history. Unfortunately in [16] the value of the fluid temperature and that of the fluid characteristics were not reported and a direct comparison was not possible. However, choosing nominal values of the fluid characteristics taken at 300 K, the wall temperature evolution (at different locations along the wall) was calculated and presented in Fig. 5, showing a good qualitative agreement with those reported in [16].

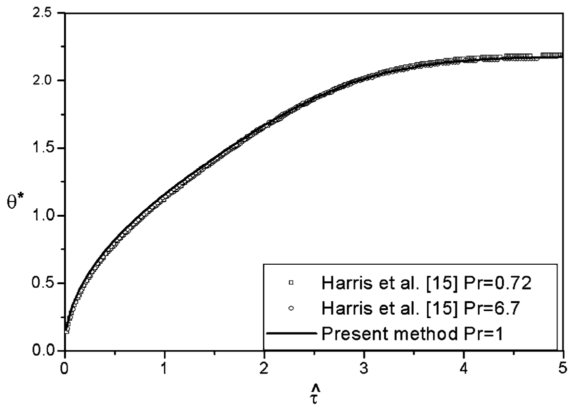

Fig. 4. Comparison among the results reported by [15] for the two cases $Pr = 0.72$ and $Pr = 6.7$ (obtained by image analysis on the scanned figure reported in the cited paper) and those obtained by the present method for $Pr = 1$ expressed in a unique form though the transformation: $\theta^* = \theta Pr^{1/3}$; $\hat{\tau} = \tau Pr^{-1/3}$.

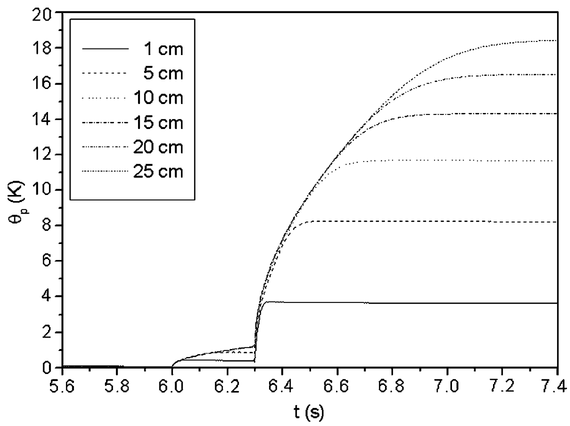

Fig. 5. Wall temperature variations at different position along the wall for comparison with the results reported by [16].

7. Conclusions

The analysis of the periodic heat transfer in forced laminar boundary layer flow over a semi-infinite plate allows to define a boundary layer "transfer function" which contains all the information about the relation between heat flux and wall temperature. This approach may prove of practical use when dealing with problems where the heat flux varies with time following an arbitrary law, as in fact the transfer function will permit to simplify the analysis and predictions: once the transfer function has been evaluated numerically, every problem characterised by uniform distribution of heat flux along the wall can be solved, without further numerical simulation, only by calculating Fourier transforms, a method certainly computationally more efficient than direct numerical simulation of the unsteady heat transfer problem. The boundary layer transfer function can be used to treat also impulsive heating problems by a suitable approximation of the heat flux time history, and a whole class of transient heating problems, namely those characterised by a uniform distribution of the heat flux along the plate, can be solved using the same function. A transformation of non-dimensional time and temperature allowed to show that, for $Pr > 0.6$, all the impulsive heating problem with a single step variation of the heat flux can be simulated by a single numerical experiment obtained, say, with $Pr = 1$. The effect of periodic heat transfer on the heat transfer coefficient was evaluated analytically for the harmonic case, showing that the effect of the unsteadiness on the average heat transfer coefficient becomes important only for relatively large frequencies, while the fluctuations of the instantaneous value of the heat transfer coefficients show a non-negligible amplitude also for lower input frequencies and low fluctuation amplitudes.

Appendix A

Consider the 1-D conduction problem set in a layer of thickness D and properties equal to those of the fluid. Let $\tilde{y}=0$ be the equation of the lower face and $d=\frac{D}{L}$ the non-dimensional layer thickness. Using the non-dimensional variables $y=\frac{\tilde{y}}{L}$; $t=\frac{\tilde{t}U_\infty}{L}$, and defining:

$$T(y,t)=T_a(y)+\int_{-\infty}^{\infty}Z(y,\omega)e^{i\omega t}\,d\omega;$$

$$q_w(t)=q_{w,a}+\int_{-\infty}^{\infty}\widetilde{\Omega}_c(\omega)e^{i\omega t}\,d\omega$$

the 1-D Fourier equation for unsteady conduction is transformed to

$$\frac{i\omega}{K}Z-\frac{\partial^2 Z}{\partial y^2}=0 \tag{25}$$

with $K=\frac{\alpha}{U_\infty L}$ and with boundary conditions:

$$\widetilde{\Omega}_c=-\frac{k}{L}\left(\frac{\partial Z}{\partial y}\right)_{y=0};\quad Z(d)=0 \tag{26}$$

Let now chose the thickness d proportional to the classical thermal boundary layer thickness, i.e., $d=m\xi^{1/2}Re_L^{-1/2}Pr^{-1/3}$, where $\xi=\frac{\tilde{x}}{L}$ and m is a pure number, introducing the transformation

$$G_c=\frac{Z}{\Omega_c}\omega^{1/2};\quad \eta=y\xi^{-1/2}Re^{1/2};\quad w=\omega\xi$$

with again $\Omega_c=\widetilde{\Omega}_c\frac{L}{k}Re_L^{-1/2}$, Eqs. (25) and (26) become

$$\begin{aligned}&iwPrG_c-\frac{\partial^2 G_c}{\partial\eta^2}=0\\&G_c(w,\eta_\infty)=0;\quad \frac{\partial G_c(w,0)}{\partial\eta}=-w^{1/2}\end{aligned} \tag{27}$$

with $\eta_\infty=d\xi^{-1/2}Re_L^{1/2}=mPr^{-1/3}$. The solution is

$$G_c=\frac{e^{\beta(\eta_\infty-\eta)}-e^{-\beta(\eta_\infty-\eta)}}{\sqrt{iPr}(e^{\beta\eta_\infty}+e^{-\beta\eta_\infty})}$$

with $\beta=\sqrt{iwPr}$. Defining $G_{c,0}(w)=G_c(w,0)$ and setting $G_c=G_{c,0}(w)\Theta_c(w,\eta)$, it is easy to find that $\Theta_c=\frac{e^{(\eta_\infty-\eta)\beta}-e^{-(\eta_\infty-\eta)\beta}}{e^{\eta_\infty\beta}-e^{-\eta_\infty\beta}}$ and $G_{c,0}(w)=\frac{w^{1/2}}{\Phi_c(w)}$ with $\Phi_c=\beta\frac{e^{\eta_\infty\beta}+e^{-\eta_\infty\beta}}{e^{\eta_\infty\beta}-e^{-\eta_\infty\beta}}$. Moreover

$$F_{c0}(\lambda)=\sqrt{Pr}G_{c0}=\frac{[\sinh(\varphi)\cosh(\varphi)+\sin(\varphi)\cos(\varphi)]+i[\sin(\varphi)\cos(\varphi)-\cosh(\varphi)\sinh(\varphi)]}{\sqrt{2}\{[\cos(\varphi)\cosh(\varphi)]^2+[\sin(\varphi)\sinh(\varphi)]^2\}}$$

where $\varphi=\eta_\infty\sqrt{\frac{wPr}{2}}=m\sqrt{\frac{\lambda}{2}}$. The asymptotic behaviour of $G_{c,0}$ when $\lambda\to 0$, is $G_{c,0r}\to\eta_\infty w^{1/2}$ and $G_{c,0i}\to 0$, thus, for the asymptotic case, $\Phi_c=\frac{1}{\eta_\infty}=\frac{Pr^{1/3}}{m}$. The asymptotic behaviour for $\lambda\to\infty$ is instead $F_{c,0r}(\lambda)\to\frac{1}{\sqrt{2}}$, and $F_{c,0i}(\lambda)\to-\frac{1}{\sqrt{2}}$.

Appendix B

Define a non-dimensional time as: $\hat{\tau}=\frac{U\tilde{t}}{\tilde{x}}Pr^{-1/3}$, from the definition of F_0 the temperature fluctuation can be written as

$$\begin{aligned}T'_w(\xi,t)&=\int_{-\infty}^{\infty}S_0(\xi,\omega)e^{i\omega t}\,d\omega\\&=\int_{-\infty}^{\infty}G_0(\xi,\omega)\Omega(\omega)\omega^{-1/2}e^{i\omega t}\,d\omega\\&=\int_{-\infty}^{\infty}\frac{F_0\Omega}{\xi^{1/2}Pr^{2/3}\lambda^{1/2}}e^{i\lambda\hat{\tau}}\,d\lambda\end{aligned}$$

Consider now a step variation in the wall heat flux, that can be written as

$$q_w(t)=\frac{q_{max}}{2}+\frac{q_{max}}{2}H(t);\quad H(t)=\begin{cases}1, & t>0\\-1, & t<0\end{cases}$$

then the function $\Omega(\omega)$ can be evaluated through the inverse Fourier transform as

$$\begin{aligned}\widetilde{\Omega}&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{q_{max}}{2}H(t)e^{-i\omega t}\,dt\\&=\frac{q_{max}Pr^{1/3}\xi}{4\pi}\int_{-\infty}^{\infty}H(\hat{\tau})e^{-i\lambda\hat{\tau}}\,d\hat{\tau}=q_{max}Pr^{1/3}\xi N(\lambda)\end{aligned}$$

Using Eq. (12)

$$\begin{aligned}T'_w(\xi,t)&=\frac{q_{max}\xi^{1/2}L}{kPr^{1/3}Re_L^{1/2}}\int_{-\infty}^{\infty}\frac{F_0(\lambda)N(\lambda)}{\lambda^{1/2}}e^{i\lambda\hat{\tau}}\,d\lambda\\&=\frac{q_{max}\xi^{1/2}L}{kPr^{1/3}Re_L^{1/2}}M(\hat{\tau})\end{aligned} \tag{28}$$

and introducing the non-dimensional temperature (see [15]) $\theta=\frac{T'}{\frac{q_{max}}{k}\left(\frac{\nu\tilde{x}}{U_\infty}\right)^{1/2}}=T'\frac{kRe_L^{1/2}}{q_{max}L\xi^{1/2}}$ Eq. (28) becomes

$$\theta^*=\theta Pr^{1/3}=M(\hat{\tau})$$

References

[1] N. Riley, Unsteady heat transfer for flow over a flat plate, J. Fluid Mech. 17 (1963) 97–104.

[2] B.T. Chao, L.S. Cheema, Unsteady heat transfer in laminar boundary-layer over a flat plate, Int. J. Heat Mass Transfer 11 (1968) 1311–1324.

[3] S.C.R. Dennis, Unsteady heat transfer for boundary-layer flow over a flat plate, in: Recent Research on Unsteady Boundary Layers, IUTAM Symposium 1971, vol. 1, Laval University Press, Que., 1972, pp. 379–403.

[4] M. Van Dyke, Computer extension of perturbation series in fluid mechanics, SIAM J. Appl. Math. 28 (1975) 720–733.

[5] R. Wang, B. Chung, L. Thomas, Transient convective heat transfer for laminar boundary layer flow with effects of wall capacitance and resistance, J. Heat Transfer 99 (1977) 513–519.

[6] M. Lachi, G. Polidori, N. Chitou, J. Padet, Theoretical modelisation in transient convective heat transfer for laminar boundary layer flow, in: Proceedings of International Centre for Heat and Mass Transfer, Turkey, 1996, pp. 27–36.

[7] B.T.F. Chung, S.A. Kassemi, Conjugate heat transfer for laminar flow over a plate with non steady temperature at the lower surface, J. Heat Transfer 102 (1980) 177–180.

[8] G. Polidori, M. Lachi, J. Padet, Unsteady convective heat transfer on a semi-infinite flat surface impulsively heated, Int. Commun. Heat Mass Transfer 25 (1) (1998) 33–42.

[9] G. Polidori, J. Padet, Transient forced convection with arbitrary variation in the wall heat flux, Heat Mass Transfer 38 (2002) 301–307.

[10] M. Rebay, J. Padet, Laminar boundary layer flow over a semi-infinite flat plate impulsively heated or cooled, Eur. Phys. J. AP 7 (1999) 263–269.

[11] M. Rebay, M. Lachi, J. Padet, Transient forced convection due to a positive step change on thermal wall conditions, Int. J. Fluid Mech. Res. 27 (2000) 468–480.

[12] G.E. Cossali, A Fourier transform based data reduction method for the evaluation of the local convective heat transfer coefficient, Int. J. Heat Mass Transfer 47 (2004) 21–30.

[13] P.T. Ireland, T.V. Jones, Detailed measurements of heat transfer on and around a pedestal in fully developed passage flow, 18th International Heat Transfer Conference Proceedings, vol. 3, 1986, pp. 975–980.

[14] R.J. Butler, J.W. Baughn, The effect of the thermal boundary conditions on transient method heat transfer measurements on a flat plate with a laminar boundary layer, J. Heat Transfer 118 (1996) 831–837.

[15] S.D. Harris, D.B. Ingham, I. Pop, Transient boundary-layer heat transfer from a flat plate subjected to a sudden change in heat flux, Eur. J. Mech. B-Fluids 20 (2001) 187–204.

[16] M. Lachi, M. Rebay, E. Cerna Mladin, J. Padet, Alternative models for transient convection heat transfer in external flows over a plate exposed to a variable heat flux, Int. J. Therm. Sci. 43 (2004) 809–816.

[17] A. Bejan, Convection Heat Transfer, John Wiley & Sons, NY, 1995.

[18] P.D. Weidman, D.G. Kubitschek, S.N. Brown, Boundary layer similarity flow driven by power-law shear, Acta Mech. 120 (1997) 199–215.

[19] E. Magyari, B. Keller, I. Pop, Heat transfer characteristics of a boundary-layer flow driven by a power-law shear over a semi-infinite plate, Int. J. Heat Mass Transfer 47 (2004) 31–34.

[20] G.E. Cossali, Similarity solutions of energy and momentum boundary layer equations for a power-law shear driven flow over a semi-infinite flat plate, Eur. J. Mech.: B-Fluids 24 (6) (2005), doi:10.1016/j.euromechflu.2005.04.007.

[21] G. Polidori, M. Rebay, J. Padet, Retour sur les résultats de la théorie de la convection forcée laminaire étabilie en écoulement de couche limite externe 2D, Int. J. Therm. Sci. 38 (1999) 398–409.